Teaching Geometry through Problem-Based Learning

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One secondary school's mathematics department decides to change its geometry curriculum to a problem- and discussion-based one.

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About seven years ago, the mathematics teachers at my secondary school came to the conclusion that they were not satisfied with our rather traditional geometry textbook. I had already begun using a problem-based approach to teaching geometry in my classes, a transition for me and my students that inspired me to write about the differences in the methodology and classroom practice (Schettino 2003).

This transition led me on a journey toward researching and learning about problem-based learning (PBL) in mathematics education at the secondary school level. My work began to intrigue my colleagues. When it came time to change our geometry textbook, instead of looking for a new textbook for the course, we decided to create our own. We followed the lead of the original writers of many of the problems I had been using (the faculty at Phillips Exeter Academy), making sure that the curriculum met the needs of the students at our independent all-girls school. Our student body has a range of educational preparation and ability, and we teach the geometry course in classes that are not "tracked." This decision led us to create our own course; it helped inspire many of us to think about teaching and learning in new ways and launched us into the new world of Problem-Based Learning (PBL).

WHAT IS PROBLEM-BASED LEARNING?

In reading about PBL, we realized that there was not one universally accepted definition for what we wanted to do. We wanted a curriculum that looked at mathematics as connected topics that would relate to one another organically and dynamically. We wanted a curriculum that empowered students and incorporated connections with prior mathematical knowledge; we wanted problems that spiraled and that would provide built-in reinforcement situated within posed problems. We also wanted a curriculum that made use of the twenty-first-century skills of communication, collaboration, and technology literacy (Partnership for Twenty-first Century Skills 2007). In addition, PBL fosters many of the skills needed for the Standards for Mathematical Practice that the Common Core State Standards call for, such as perseverance in problem solving and developing abstract and quantitative reasoning skills (Common Core State Standards Initiative 2010). Further, PBL provides for diversity in learning and potential for using more alternative pedagogical approaches in the future. It stresses the value of discourse and allows us to consider problems from multiple perspectives. Without a textbook that might "specify the appropriate tool to be used for the given problem," we make room for the "crucial ethical moment of reflecting on whether the means suit the ends" (de Freitas 2008).

I found many studies showing that PBL allowed students to attain equal or greater achievement on standardized testing as students taught by direct instruction (Savery 2006) and to do better on problem-solving and long-term knowledge retention (Strobel and van Barneveld 2009). One study even showed that PBL is more effective with lower-ability students (Ridlon 2009).

In general, *problem-based learning* is defined as "an instructional (and curricular) learner-centered approach that empowers learners to … integrate theory and practice and apply knowledge and skills to develop a viable solution to a defined problem" facilitated by a teacher who "guides the learning process and conducts a thorough debriefing at the conclusion of the learning experience" (Savery 2006).

However, I define *problem-based learning* as follows: an instructional approach of curriculum and pedagogy where student learning and content material is constructed (and co-constructed) through the use, facilitation, and experience of contextual problems in a decompartmentalized, threaded topic format in a discussion-based classroom setting where student voice, experience, and prior knowledge are valued.

Our curriculum begins with multiple topics running in parallel, and these threads introduce new concepts through scaffolded problems. Because some problems are theoretical in nature (e.g., connecting the Pythagorean theorem and the distance formula), not all problems are situated in real-life contexts, so our definition differs from a more project-based curriculum.

The problems are designed so that homework assignments motivate class discussion for the next day. The teacher must plan deliberately, choosing specific problems to lead into future problems sequentially. Each homework assignment consists of seven or eight problems—perhaps fewer early in the year.

This new type of "homework," where full, correct answers are not always expected, serves many purposes:

- It provides an opportunity for students to review material from past courses.
- It triggers prior knowledge for an upcoming problem.
- It inspires construction of new knowledge.
- It introduces new terminology.
- It allows students to practice a new skill.
- It challenges the more able students (differentiated instruction).
- It enables students to see the same new idea represented differently.

PBL homework is very different from homework in a traditional, direct-instruction class. For all the reasons listed above, it is important for students to form a different understanding of what the purpose of homework is. Teachers must send the message that they value risk taking and intuition even when students may not have a complete and full answer, so that construction of knowledge can happen in a safe and open environment (see Saphier and Gower 1997). Teachers must facilitate discussion by pushing for explanations, revoicing comments, summarizing conclusions, evaluating hypotheses, and encouraging multiple representations (Hmelo-Silver and Barrows 2006).

THE PBL CURRICULUM STRUCTURE

Before implementing the change, the mathematics department created a curriculum map of the geometry course and of the prerequisites for the next course. We also created a list of assumed prior knowledge and skills and began writing the problems, ensuring a logical order. Problems that reviewed a key skill that would help students on an upcoming problem or that motivated a new idea were occasionally included.

For example, to lead to the idea of the area of sectors of circles, students needed to be able to conceptualize the partitioning of the area of the circle. The formula for the area of a circle is something students generally have seen in middle school mathematics courses, but they needed a brief review. So, after introducing the concept of a central angle in a problem and reviewing circumference and area, we wrote this problem:

> If the central angle of a slice of pizza is 36°, how many pieces are in the pizza?

The resulting conversation allows students to construct the knowledge that the central angle of the sector (the piece of pizza) directly determines the fractional part of the whole circle (the whole pizza). The teacher then must facilitate the generalization of this conjecture. For example, some students assume that the pizza has ten slices. When pressed for a reason, they may respond that they divided 360°by 36 without further clarification. If asked to consider a central angle of 24°, the answer 15 might not come as quickly. Students need to hear others' ideas to articulate the concept that the number of degrees in the central angle is a factor of 360. When another student explicitly states that 36 physically fits into 360 ten times, the connection is made.

With PBL, a student's solution often includes either a physical or a virtual model of a pizza. Because this question was posed in a problem, the more visual learners were able to be the agents of their own learning and find their way of understanding the concept of the central angle. As they share their unique ideas with the class, they gain confidence and authority in their knowledge construction.

An upcoming topic in the curriculum extends the central angle idea to a proportional algorithm for arc length and then sector area. Following the central angle question, I make a point to ask the class a scaffolding question: "If there are ten pieces of pizza, *how much* of the pizza is one of the pieces?" Most students can say that each piece of the pizza is one-tenth of the whole pizza. I ask this question to reinforce the concept that the central angle's relationship to the total sum of 360 is going to be important.

The pizza analogy lays the groundwork for what follows. Recalling the proportional division algorithm that students had created with my scaffolding question, we introduce the concept of arc length through this problem:

> A 12-inch pizza is evenly divided into 8 pieces. What is the length of the crust of one piece?

This problem creates many questions in the students' minds. What does the 12 inches describe—the radius? The diameter? Is the pizza circular? These are all wonderful questions that generate class discussion. Some students make a connection—they realize that they can extend proportionality to the circumference (i.e., the crust)—and come to class prepared to discuss this idea. Others, using their prior knowledge and assumptions about what the 12 inches represent, find only the circumference (the length of the entire crust). Once the connection to the previous problem and the fraction of the entire circle is made, students realize that they can take the fraction of the length of the whole crust. Eventually, they generalize this concept into a formula—ideally, a student initiates a discussion, but, if not, I will question in a way that scaffolds the student discussion to the formula

$$\text{arc length} = \frac{\text{central angle}}{360°} \bullet 2\pi r.$$

The hope is that students will be inspired to extend their idea from arc length to sector area. However, if they do not do so spontaneously, subsequent problems move these topics forward.

A typical class begins with student problem presentations. Students can present their work in various ways. They may volunteer or be assigned; they may work at the blackboard or with technology; they may work individually or in pairs. The presenting student is given time to explain her initial thought process and then is given time for additions or corrections. The teacher facilitates the discussion concerning the correctness of the solution, connections to prior knowledge, and new ideas that stem from that problem.

The students must have the freedom to explore their ideas while using their own voices, thus gaining ownership of the material. The teacher should attempt to balance respectful intervention with presentation of new material and elicitation of information from the students.

If the problem's objective is not generated by the class discussion, the teacher must find a less natural way to introduce it. Most important, the teacher must conduct a thorough debriefing at the conclusion of the problem to make sure students have understood the method presented and how to apply it in the future.

PBL IN ACTION

An advantage of this type of curriculum is that students have opportunities to make connections themselves. At one point in the curriculum, multiple threads occur simultaneously—dilations as transformations, a review of ratios and proportions, similar triangles, and an introduction to trigonometry.

In a problem introducing the sine ratio, students are asked to construct, using a ruler and a protractor, a right triangle with a hypotenuse of 15 cm and an angle of 27°. They are then asked to measure the side opposite the constructed angle as accurately as possible and find what percentage of the hypotenuse that side is. During the class discussion, students compared their answers and found that they were all about the same—around 45%. This result allowed me to introduce the sine function.

One student commented that she had done the problem incorrectly because she had misread the directions and had made the hypotenuse of her triangle 10 cm instead of 15 cm. "Why did I still get the right answer?" she asked. I turned that question to the class for discussion. After some thought, one student offered, "Well, all we were looking at was how the sides related to each other. There was a 27° angle and a 90° angle, so she just made a similar triangle like in that problem we did yesterday." This student was making the connections between similar triangles and the sine function. A PBL curriculum enables students to make such connections and see overarching relationships.

Another way to see PBL in action is through collaboration. In a conversation about finding the area of parallelograms before students were given an explicit formula, one student was reluctantly presenting her solution to a problem. The question was to find the area of parallelogram *ABCD* given its vertices, as shown in **figure 1.**

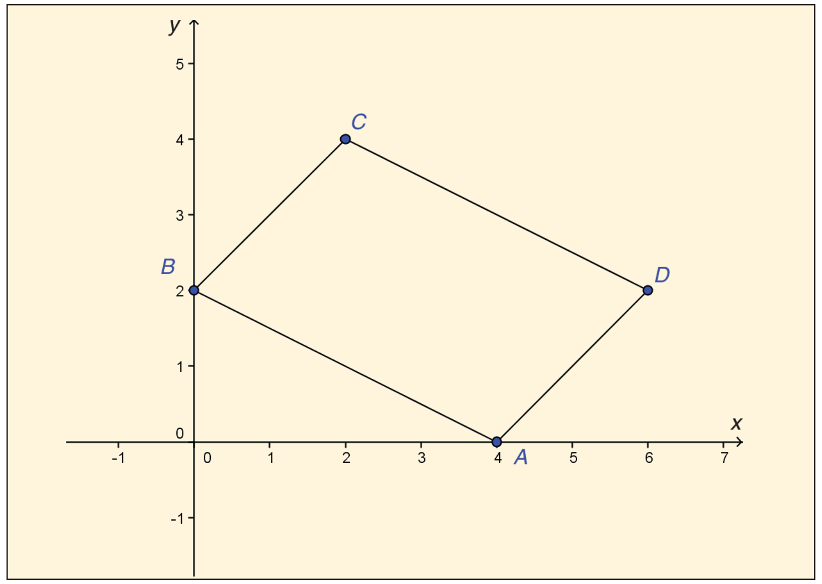

Fig. 1 Stephanie confused side lengths with base and height.

Teacher: [*addressing student who is doing presentation*] OK, Stephanie, so what did you do?

Stephanie: OK, uh, well, I thought that because, like, a quadrilateral would be, like, base times height, so I just took the length, and I got the distance from *A* to *B* and then the distance from *A* to *D,* and then I timesed them together…

Annie: I definitely did not do that …

Laura: Yeah.
Stephanie: But I'm not sure if you needed to get the altitude?
Meghan: Oh, that's what I did …
Teacher: How is that different from what Annie just said in number 9? What's the difference between just multiplying AD and AB together … —the *sides* together—and multiplying base times height?

In this discussion, Stephanie shares her (incorrect) conjecture about how to find the area of the parallelogram and has heard the responses of some of her classmates. Their responses prompt her to question her method. She thinks about what was done in a previous problem where students calculated area by finding an altitude algebraically. After receiving the feedback, Stephanie corrects her mistake and rethinks the problem. Although Annie and Leah think that Stephanie's idea of how to find the area of the parallelogram was incorrect, the teacher tries to allow them to decide which method was correct by comparing Stephanie's with one that Annie had used in a previous problem. By comparing different methods side-by-side and having students present their ideas to the class, the teacher can allow students' prior knowledge and conjecture to reach a conclusion about the correct way to find the area of the parallelogram.

After Stephanie finished, another student, Becca, wondered whether her own approach was incorrect. She proceeded to show her diagram (see **fig. 2**). Becca described her approach: She drew a rectangle around the parallelogram, found the area of the four right triangles at the corners, and subtracted the four triangular areas from the large rectangle, resulting in the correct area.

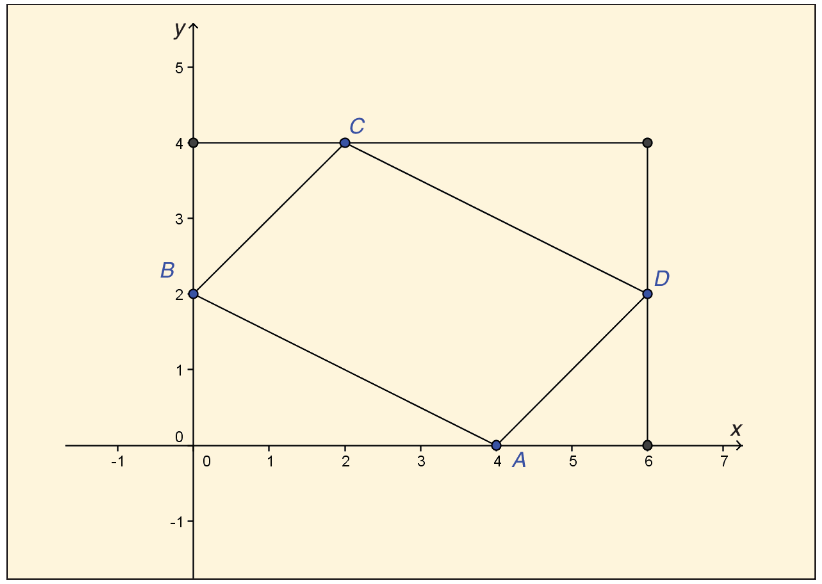

Fig. 2 Becca was able to enclose the parallelogram in a rectangle and decompose the rectangle's area.

Becca's geometric approach for finding the area was very different from Stephanie's algebraic approach, which entailed writing equations of altitudes and using the distance formula to find the necessary lengths. However, at the end of the discussion of this problem, all the students had an opinion about which method they appreciated more. They had been engaged and interested in seeing which would work better for them.

The PBL curriculum gives students the opportunity to create their own solution methods and the freedom to express their own ideas; it also shows them the value of risk taking and efficiency in problem solving. However, it is the teacher's responsibility to summarize the methods and lay out the advantages and disadvantages of each so that students can feel empowered to make those problem-solving decisions for themselves.

MOVING FORWARD

Our anecdotal research has found that this type of curriculum has met these goals. Reflecting on the effects of this curricular change, one student said:

> You can see people … asking you a question wherever you are in math class … You can see that everyone here wants to figure out how they got a certain problem. There's more of an interest than just getting something right, they want to understand how.

We still need further evidence to support achievement and retention of material for sequential courses.

The benefits to student learning in mathematics that we have observed—independence in problem solving, improved communication, empowerment in student voice, and agency in learning—far outweigh the traditional concerns that many teachers feared. This is not to say that there were no obstacles (see Schettino [2003] for more on obstacles in transitioning). However, teachers working together with open minds, communicating in professional development settings, and visiting classrooms have allowed this mathematics department to find a balance in this transition. As a result, we have met our goals for teaching geometry that are consistent with both the NCTM Standards, the Common Core State Standards, and our hopes for the future of our students as independent problem solvers.

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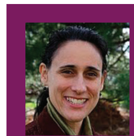

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