Developing understanding of parametric equations:

1. I have been observing the motion of a bug on my graph paper. When I started watching, it was at the point (1, 2). Ten seconds later it was at (3, 5). Another ten seconds later it was at (5, 8). After another ten seconds it was at (7, 11).

(a) Draw a picture that illustrates what is happening.

(b) Write a description of any pattern that you notice. What assumptions are you making?

(c) Where was the bug 24 seconds after I started watching it?

(d) Describe the relationship between the number of seconds the bug has been moving and the bug's location on my graph paper.

(e) When looking at a graph of the bug's motion, is there any way to denote the value of the time on the graph? In other words, is there a way to signify the variable t on the x-y plane?

2. From its initial position at (3,4), an object moves linearly, reaching (9,8) after two seconds and (15,12) after four seconds.

(a) Predict the position of the object after six seconds; after nine seconds; after *t* seconds.
(b) Is there a time when the object is equidistant from the coordinate axes? If so, where is it?

3. The *x*- and *y*-coordinates of a point are given by the equations shown at right. $\begin{array}{l} x = 2 + 2t \\ y = 5 - t \end{array}$. The position of the point depends on the value assigned to *t*. Use your graph paper to plot points corresponding to the values *t* = –4, –3, –2, –1, 0, 1, 2, 3, and 4. Do you recognize any patterns? Describe them.

4. (Continuation) Plot the following points on the coordinate plane: (1,2), (2,5), (3,8). Write equations, similar to those in the preceding exercise, that produce these points when *t*-values are assigned. There is more than one correct answer.

5. In a dream, Blair is confined to a coordinate plane, moving along a straight line with a constant speed. Blair's position at 4 a.m. is (2,5) and at 6 a.m. it is (6,3). What is Blair's position at 8:15 a.m. when the alarm goes off?

6. At noon one day, Corey decided to follow a straight course in a motor boat. After one hour of making no turns and traveling at a steady rate, the boat was 6 miles east and 8 miles north of its point of departure. What was Corey's position at two o'clock? How far had Corey traveled? What was Corey's speed?

7. (Continuation) Assuming that the gas tank initially held 12 gallons of fuel, and that the boat gets 4 miles to the gallon, how far did Corey get before running out of fuel? When did this happen? How did Corey describe the boat's position to the Coast Guard when radioing for help?

8. The x-component of a velocity vector is 15 mph, and the speed is 17 mph. Find all possible values for the y-component.

9. After being dropped from the top of a tall building, the height of an object is described by *y=400-16t^2* , where *y* is measured in feet and *t* is measured in seconds.

(a) How many seconds did it take for the object to reach the ground, where *y* = 0?
(b) How high is the projectile when *t* = 2, and (approximately) how fast is it falling?

10. After being thrown from the top of a tall building, the position of a projectile is described parametrically by (x,y) = (48t, 400-16t^2) where *x* and *y* are in feet and *t* is in seconds.

(a) How many seconds did it take for the projectile to reach the ground, where *y* = 0? How far from the building did the projectile land?
(b) How fast was the projectile moving at *t* = 0 when it was thrown?
(c) Where is the projectile when *t* = 2, and (approximately) how fast is it moving?

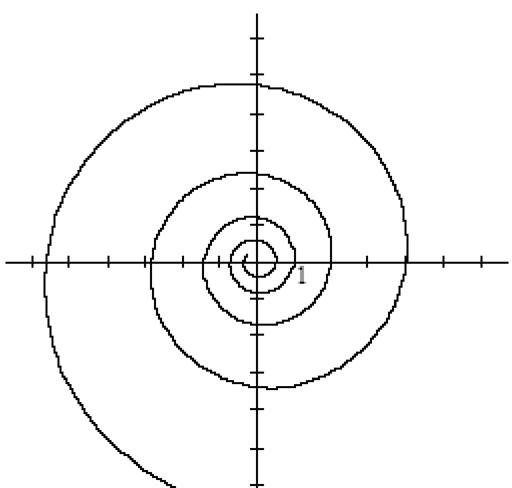
11. After being thrown from the top of a tall building, a projectile's path is described parametrically by $(x, y) = (60t, 784 - 16t^2)$, where x and y are in feet and t is in seconds. The sun is directly overhead, so that the projectile casts a moving shadow on the ground beneath it.
    a. Create a graph of this motion for t > 0 on your graphing calculator.
    b. At t = 1, how fast is the shadow moving?
    c. At t = 1, how fast is the projectile losing altitude?
    d. Approximately, how fast is the projectile moving?

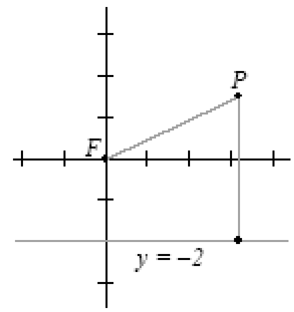
12. Parametric equations (x,y) = (t-sint, 1-cost) trace out an interesting curve. Make a sketch of this graph that is called a *cycloid.* Consider the point on this curve where $t = \pi$. What are the coordinates of this point? Using an approximation method, what is the approximate slope of the curve at this point? Remember that slope, in the Cartesian plane, even for a parametrically defined function is still $\frac{\Delta y}{\Delta x}$. Do the same at the point where $t = 2\pi$.

Exploring more advanced polar graphs

1. Spirals are fundamental curves, but awkward to describe using only the Cartesian coordinates *x* and

y. The example shown at right, on the other hand, is easily described with polar coordinates – all its points fit the equation/ $r = 2^{\frac{\vartheta}{360}}$ (using degree mode). Choose three specific points in the diagram and make calculations that confirm this. What range of ϑ -values does the graph represent?





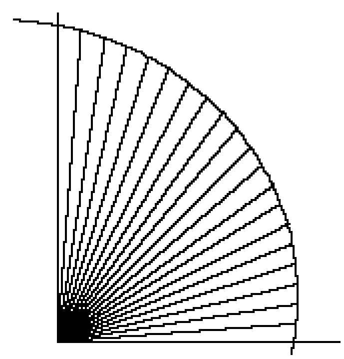
2. Let the focal point *F* be at the origin, the horizontal line *y* = –2 be the directrix, and $P=(r,\vartheta)$ be equidistant from the focus and the directrix. *Using the polar variables r and* ϑ , write an equation that says that the distance from *P* to the directrix equals the distance from *P* to *F*. The configuration of all such *P* is a familiar curve; make a rough sketch of it. Then rearrange your equation so that it becomes $r=\dfrac{2}{1-\sin\vartheta}$, put your calculator into polar mode, and graph this familiar curve. On which polar ray does no point appear?

3. Let *F* be the focal point (0,0), the horizontal line *y* = –12 be the directrix, and *P* be a generic point in the plane. Using the polar variables *r* and ϑ , write an equation that says that the distance from *P* to *F* is *half* the distance from *P* to the directrix. For example, you should find that the coordinates *r* = 4.8 and $\vartheta = 210$ degrees describe such a *P*. The configuration of *all* such points *P* is a familiar curve. After you make a rough sketch of it, put your calculator into polar mode and graph this equation. (First rearrange your equation so that *r* is expressed as a function of ϑ .) What type of curve is this?

4. Given a positive number other than 1, the polar equation $r=b^{\frac{\vartheta}{360}}$ represents a *logarithmic spiral*. Such a graph crosses the positive *x*-axis infinitely many times. What can be said about the sequence of crossings? What about the intercepts on the negative *x*-axis? What if *b* is less than 1? Examine these questions using the examples *b* = 3, *b* = 1.25, *b* = 0.8, and *b* = 13.

5. Working in degree mode, graph the polar equation $r = 4\cos\vartheta$ for ϑ -values between 0 and 180. Identify the configuration you see. Find Cartesian coordinates for the point that corresponds to $\vartheta = 180$.

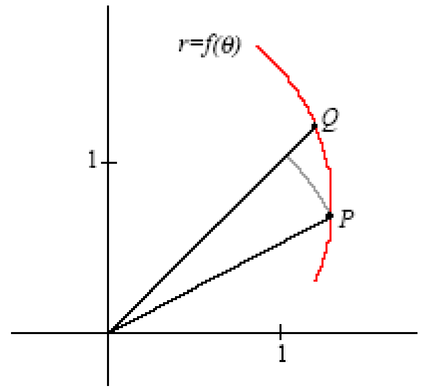
Developing understanding of polar area and arclength



1. The diagram at right shows the portion of the spiral $r=1.2^{\theta}$ that corresponds to $0\le\theta\le\dfrac{\pi}{2}$. (You might want to graph it on your graphing calculator so you can see how much of the spiral you are looking at in this picture). The area enclosed by this arc and the rays *r* = 0 and $r=\dfrac{\pi}{2}$ can be found by integration.
   a. First imagine cutting the region into many small narrow sectors (the diagram shows only twenty), each of which **nearly** has a constant radius. Justify this statement.
   b. Now consider only one of these extremely narrow sectors. Use your knowledge of circular sector area from geometry to find an expression for the area of one of these regions.

c. Why is the area of the whole region equal to the value of the definite integral $\int_0^{\frac{\pi}{2}} \frac{1}{2}(1.2)^{2\theta} d\theta$? Justify each part of the integral, including the function, the differential and the limits of integration.

2. To find the length of a polar curve $r = f(\theta)$, the diagram at right is helpful. Place the labels *r*, $\Delta\theta$, $r\Delta\theta$, and Δr where they belong in the diagram, then deduce an approximation formula for the length of arc *PQ*. Convince yourself that the accuracy of this approximation improves as $\Delta\theta$ becomes very small. Then do an example: Show that the length of the cardioids $r = 1 + \cos\theta$ is given by the definite integral $\int_0^{2\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\, d\theta$.
Finish the job by evaluating the integral.



Motivating MacLaurin Polynomials

1. Of all the straight lines that can be drawn through the point (0,1), the one that best fits the graph of y=cosx is of course the tangent line y=1. It is inviting you to consider the problem of finding the parabola that best fits the curve y=cosx at the point (0,1). With this in mind, find the numbers a ,b, and c that make the quadratic function $f(x) = ax^2 + bx + c$ and its first two derivatives agree at x=0 with the cosine function and its first two derivatives. Graph both functions on your calculator. Could you have anticipated what you see?

2. Sketch the graph of $f(x) = \ln(x+1)$ for −1 < x, noticing that the curve goes through the origin. What is the slope of the line that is tangent to the curve at the origin? What quadratic polynomial $p(x) = a_0 + a_1x + a_2x^2$ best fits the curve at the origin? What does it mean for a polynomial to "best fit" a transcendental function? What would be the requirements? With those in mind, what cubic best fits?

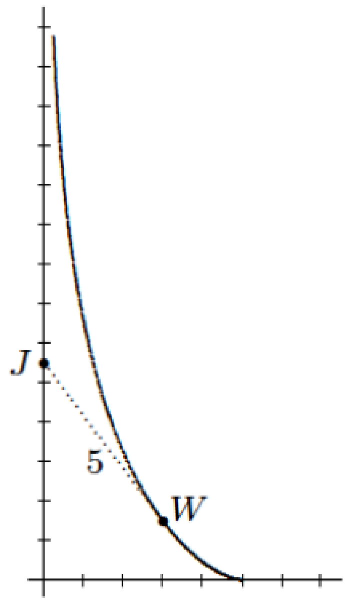
3. Find an equation for the fifth-degree polynomial p(x) that has the following properties: $p(0) = 0$, $p'(0) = 1$, $p''(0) = 0$, $p'''(0) = -1$, $p^4(0) = 0$ and $p^5(0) = 1$. How would these derivatives affect the formation of $p(x)$?

4. Given a general polynomial $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, it is routine to verify that $p(0) = a_0$, $p'(0) = a_1$ and $p''(0) = 2a_2$. What about $p'''(0)$ and the fourth derivative at zero? In general, what is the value of the kth derivative of p evaluated at zero?

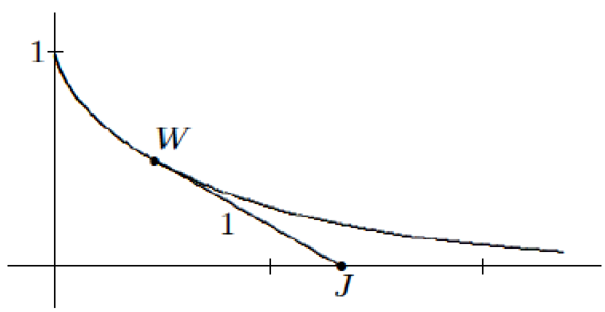
5. Given a differentiable function *f*, you have seen how to use the values $f(0)$, $f'(0)$, $f''(0)$ … to create polynomials that approximate *f* near x = 0. The coefficients of these polynomials are calculated using values of the derivatives of *f*. Namely, the coefficient of x^n is $a_n = \frac{1}{n!} f^n(0)$. Use this recipe to calculate the sixth

Maclaurin polynomial $P_6(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6$ for the cosine function, and graph both the cosine function and the polynomial for $-4 \le x \le 4$.

Developing Chase Problems with Differential Equations Geometrically

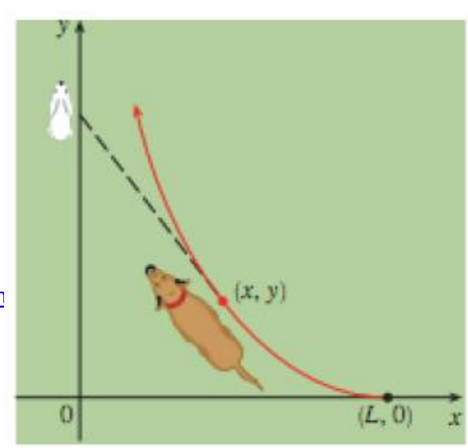


1. Starting at the origin, Jamie walks along the positive y-axis while holding a 5-foot rope (which appears dotted in the diagram) that is tied to a wagon. The wagon is initially at (5,0) and its path is shown at right as Jamie moves up the y-axis. The curve is an example of a tractrix.
    a. The path of the wagon includes the point whose coordinates are approximately (3.1, 1.493). Where is Jamie when the wagon reaches this point?
    b. Explain how the differential equation $\frac{dy}{dx}\frac{-\sqrt{25-x^2}}{x}$ defines this tractrix.
    c. Starting at (5,0) apply a 4-step Euler's method to find the y value of the wagon when x=3. Explain why you will need to use negative values of Δx. Also, explain why you can be sure that your answer will be less than 1.493.



2.
3.
4.
5.

2. Now Jamie is walking along the x-axis and pulling a wagon with a rope of unit length. The wagon is initially at the point (0,1). So, W=(x,y) rolls along the path of the tractrix. Jamie walks in such a way that the speed of W is 1 unit per second.
    a. Thinking parametrically and geometrically, how can you express the vertical velocity component, $\frac{dy}{dt}$, of the wagon in terms of y? Explain your thinking.
    b. Do the same for the horizontal velocity component, $\frac{dx}{dt}$, of the wagon in terms of y? Explain this.
    c. Show that $y = ke^{-t}$ for some positive constant k. What is the interpretation of the constant k?



3. A dog sees a rabbit running in a straight line across an open field and gives chase. In a rectangular coordinate system, as shown in the figure, assume the following:
    a. The rabbit is at the origin and the dog is at the point (L,0) when it first sights the rabbit.

b. The rabbit runs straight up the y-axis and the dog always runs straight for the rabbit.
c. The dog runs at the same speed as the rabbit.

A. Show that the dog's path is the graph of the function $y = f(x)$ where y satisfies the differential equation $x\frac{d^2y}{dx^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$.

1. Let's let (0,S) be the coordinates of the rabbit on the y-axis and P=the length of the dog's path along the curve, so $\frac{ds}{dx} = \frac{dP}{dx}$ in this problem. Why?

2.Write an integral expression using arclength for the length of the path of the dog (starting at a point (L,0) and ending at point (x,0)) – important hint – you must consider the dog's direction here.)

3. Give this integral expression, what is an expression for $\frac{dP}{dx}$, the dog's speed along the path?

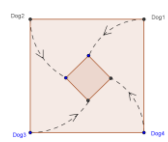
4. Write an expression for the slope of the tangent line $\frac{dy}{dx}$ at the point (x,y), the dog's current position along P. Be sure to include the point (0,S) in this expression.

5. Solve this expression for S, and using the product rule, find an expression for $\frac{dS}{dx}$.So since we know that the rabbit and the dog are moving at the same rate, how can we justify that second order differential equation now?

B. Determine the solution of the equation in part A that satisfies the initial conditions given where $y = y' = 0$ for the dog. (Hint: use the substitution $z = \frac{dy}{dx}$).

C. Does the dog ever catch the rabbit?

4. Four dogs start at the corners of a square of side a, each dog chases the one counterclockwise from it. If they all start at the same time and run at the same speed, how far has each dog run by the time they all collide at the center of the square?



a. Imagine zooming in infinitely close at one of the dogs at a corner of that shrinking square. Let's call the distance from the center r, and think in polar coordinates. Label dr , $rd\theta$, $r+dr$ in the diagram on the next page and justify your decisions. What are all these functions of (i.e. what is the real independent variable in this problem?)

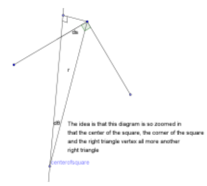
b. Why can we claim that the triangle on the exterior portion of the square is an isosceles triangle? (Remember that we are working with infinitesimally small distances here).

c. Justify the expression $\tan\left(\frac{\pi}{4}\right) = \frac{-dr}{rd\theta}$.

d. Why does $\frac{dr}{d\theta} = -r$? Solve this differential equation.

e. Why is it true that when $\theta = 0$, r=$\frac{a}{\sqrt{2}}$?

f. Using that information, solve for the constant of integration in your solution to the differential equation.



g. So now all we need to do is find out how far the dog has run when it collides with the others at the center of the square. That happens at r=0, which means that $\theta \to \infty$ (can you see why?). We need to find the arclength of the dog's path, which is given by the differentials, ds. Why can we write $ds = \sqrt{2}rd\theta$?

h. Justify why the dogs path is given by the improper integral $s = \int_0^\infty ds = \int_0^\infty \sqrt{2}rd\theta$. Using our equivalent expression for r, solve this improper integral to find the length of the dog's path!