Motivational Problems Sequences and Series

1. On 1 July 1998, you deposit 1000 dollars into an account that pays 6 percent interest annually. How much is this investment worth on 1 July 2018? Before you turn your calculator on, write an expression for the answer that you could easily put into your calculator.

2. What if you wanted to add up the first 10 perfect squares? How would you go about doing it? What is you wanted to add up the first 50 perfect squares? How long would it take you to write them all out? Put them in your calculator? Try it.

3. If a sequence can be easily written in a sequence formula, it is a matter of notation to write out the *series* that represents the sum of all the terms in the sequence. Mathematicians use the greek letter Sigma, $\sum$, which is analogous to a capital S in the English language (representing the word "Sum"). For example, if you wanted to write the series 1+2+3+….+100 you would need a shorthand way of writing all 100 terms at once. We write $\sum_{k=1}^{100} k = 1 + 2 + ... + 100$. We call the *k* the index of summation, as it counts how many terms you are adding together. The other terms are as follows: $\sum_{k=\text{lower bound}}^{\text{upper bound}} \text{sequence formula in terms of } k$. Write the following sequences in this *summation notation*:

a. 2+4+6+8+…..+100 b. 81+54+……+$10\frac{2}{3}$ c. 81+69+57+…+-51

4. (Continuation from above) On 1 July 1999, you deposit 1000 dollars into an account that pays 6 percent interest annually. How much is *this* investment worth on 1 July 2018? Answer the same question for deposits made on 1 July 2000, 1 July 2001, and so forth, until you see a pattern developing in your calculator-ready expressions.

6. Suppose each year a child is born in the Willard family and for that child the parents deposit 1000 dollars into an account on 1 July *that* year. So the parents create for each child an account with a deposit of $1000 in it growing at an annual interest rate of 6%. After 20 children, how much have the parents accumulated in all of their children's accounts?

The problem is now to calculate the combined value of *all* these deposits on 1 July 2018, including the deposit made on that final day. Rather than getting the answer by tediously adding the results of *twenty one* separate (but similar) calculations, we can find a shorter way. Let *V* stand for the number we seek, and observe that $V = 1000(1.06)^0 + 1000(1.06)^1 + 1000(1.06)^2 + ... + 1000(1.06)^{19} + 1000(1.06)^{20}$ is the very calculation that we wish to avoid. To help us simplify things, obtain a second equation by multiplying both sides of this equation by 1.06, then find a way of combining the two equations to obtain a compact, easy-to-calculate formula for V.

7. (Continuation) You have seen that any list *first, first*multiplier, first*multiplier* ^{2}… in which each term is obtained by multiplying its predecessor by a fixed number, is called a geometric *sequence*. A geometric *series*, on the other hand, is an *addition problem* formed by taking consecutive terms from some geometric sequence. Two examples: 16 + 24 + 36 + 54 is a four-term geometric series whose sum is 130, and 32 –16 +8 – 4 + + 0.125 L is a nine-term geometric

series whose sum is 21.375. Consider now the typical geometric series, which looks like $t_1, t_1 \cdot r, t_1 \cdot r^2, ... t_1 \cdot r^n$. Find a compact, easy-to-calculate formula for the sum of all these terms.

8. A similar formula can be derived for the finite sum of an arithmetic series. Suppose there was a sum $S_n = \sum_{k=0}^{n} t_1 + kd$. Write out the first 4 terms of this series.

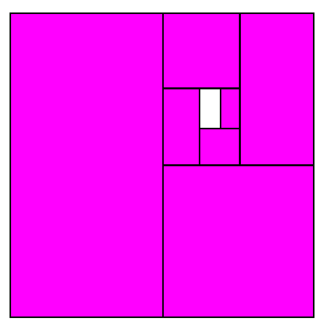
9. (continuation) Try to come up with this formula yourself! (In other words, don't look in the book!) You can write out the arithmetic series S_n as $S_n = t_1 + (t_1 + d) + (t_1 + 2d) + ... + (t_1 + (n-1)d) + (t_1 + nd)$. Now look at the last term as t_n, then the next to last term would be $t_n - d$ Now rewrite this sum in reverse order with t_n coming first and t_1, and then <u>add the two equations</u>. You should have an expression for $2 \cdot S_n$. With a little algebraic manipulation, you should be able to derive a formula for S_n. Check your answer with the work on p.487 on your text!

10. The third term of a geometric sequence is 40, and the sixth term is 135. What is the seventh?

11. A speckled green superball has a 75% rebound ratio. When you drop it from a height of 16 feet, it bounces and bounces and bounces ...
(a) How high does the ball bounce after it strikes the ground for the third time?
(b) How high does the ball bounce after it strikes the ground for the seventeenth time?
(c) When it strikes the ground for the second time, the ball has traveled a total of 28 feet in a *downward* direction. Verify this. How far downward has the ball traveled when it strikes the ground for the seventeenth time?

12. (Continuation) At the top of its second rebound, the ball has traveled 21 feet *upwar*d.
(a) At the top of its seventeenth rebound, how far upward has the ball traveled?
(b) When you catch the ball at the top of its seventeenth rebound, how far has it traveled?
(c) How far would the ball travel if you just let it bounce and bounce and bounce... ?

13. What does the figure at right suggest to you about the geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...$

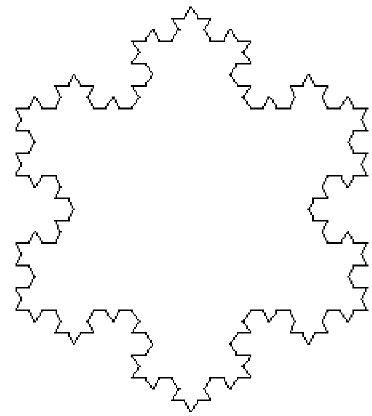


14. The repeating decimal 0.12121212... can be thought of as an infinite geometric series by seeing it as the sum of an infinite amount of finite decimals like so $0.12 + 0.0012 + 0.000012 + ...$ Write it in the form $s_1 + s_1 \cdot r + s_1 \cdot r^2 + ...$ Express the series using sigma notation, and then find its sum. What is the rational number equivalent to 0.12121212...?

15. Write down the first few terms of any geometric sequence of positive terms. Make a new list by writing down the logarithms of these terms. What kind of sequence is this now? Justify your answer.

16. Suppose that r is a number strictly between –1 and 1. What can be said about the value of r_n when n is a large positive integer? What can be said about $\frac{a - ar^{n+1}}{1-r}$ when n is a large positive integer? Give an example of an infinite geometric series whose sum is $\frac{12}{1-\left(\frac{1}{3}\right)}$.

17. The *Koch snowflake* we previously studied is an example of a fractal curve of infinite length. As suggested by the figure, however, the area enclosed by this curve is finite.
Suppose that the area enclosed by stage 0 (the initial equilateral triangle) is 1. How many triangles are added at stage 1 and how much area do they add at stage 1? Do the same for stage 2.
Show that the area enclosed by the completed snowflake can be obtained with the help of a geometric series. It might be helpful to remember that when triangles are similar and have a scale factor of $a:b$ the ratio of their areas is $a^2 : b^2$



18. Express the geometric series $28 + 16.8 + ... + 28(0.6)^{15}$ in sigma notation, and find its sum.

19. PROJECT- (quiz grade) *Repaying loans*.
The bank has just granted Miller a $10000 loan, which will be paid back in 48 equal monthly installments, each of which includes a 1% interest charge on the unpaid balance. The bank's loan officer was amazed that Miller (who knows all about geometric series) had already calculated the correct monthly payment. Here is how Miller figured it out:

(a) Pretend first that the monthly payments are all $300. The first payment must include $100 just for *interest* on the $10000 owed. The other $200 *reduces the debt*. That leaves a debt of $9800 after the first payment. Follow this line of reasoning and calculate the amount owed after four more payments of $300 have been made.
(b) Now introduce some notational shorthand: Let *A n* be the amount owed after *n* payments (so that *A* 0 = 10000), let $r = 0.01$ be the interest rate, and let *P* be the monthly payment (which is not 300). Explain why $A_1 = A_0 - (P - rA_0) = (1+r)A_0 - P$ *and* $A_2 = (1+r)A_1 - P$ then write a general recursive equation that expresses A_n in terms of A_{n-1}.
(c) Apply the recursive equation to express each A_n in terms of A_0. For example, you can write $A_2 = (1+r)A_1 - P = (1+r)[(1+r)A_0 - P] - P$ so that $A_2 = (1+r)^2 A_0 - (1+r)P - P$.
You should see a pattern developing. It involves the finite geometric series $P + (1+r)P + (1+r)^2 P + ... + (1+r)^{n-1} P$
(d) Explain why $A_{48} = 0$ for Miller's loan. Then set $A_n = 0$ in your answer to (c), and solve for P. This expresses the monthly payment in terms of A_0 (which is 10000), r (which is 0.01), and n (which is 48). In the case of Miller's loan, the monthly payment is less than $300.